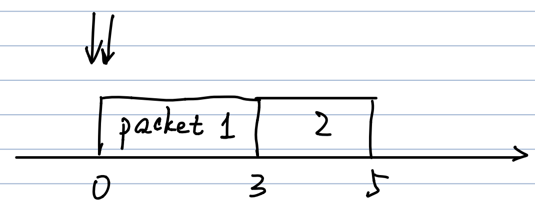
Lecture 16

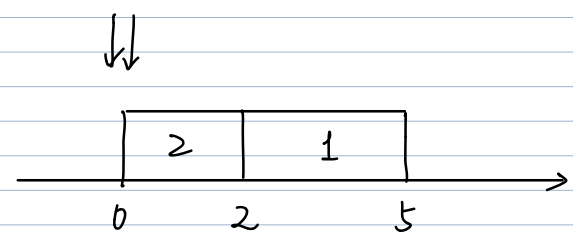
Age-optimal Scheduling in Queues. (3)

Sample-path methods:

① Interchange arguments:



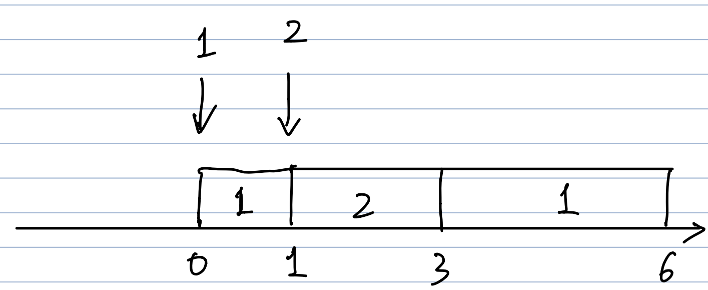
Total delay: $3 + 5 = 8$



Total delay: $2 + 5 = 7$

The total delay reduces by interchanged the two packets.

stachastic task arrivals.



X_1: service time of packet $1 = 4$ seconds.

X_2: - - - - - - - - — $2 = 2$ seconds.

idea: compare the remaining processing time of the packets.

optimal policy for minimizing total delay / average delay:

preemptive SRPT (shortest remaining processing time first).

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② forward induction:

Define system state. $\vec{V}_\pi(t)$, and partial order $\leq$.

goal: prove that, on every sample path.

$$\vec{V}_P(t) \leq \vec{V}_\pi(t), \quad \forall t, \forall \pi \in \Pi.$$

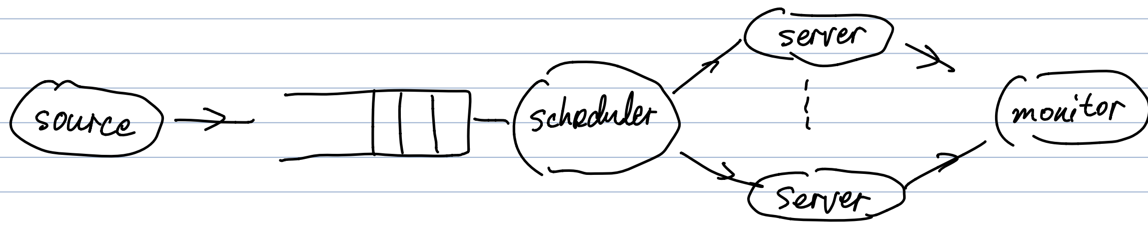
Reading: SRPT 1968. 1978.

HW3: Problem 1.

Briefly explain the forward induction method for proving the optimality of preemptive SRPT. Explain, in the proof:

① what is the system state?

② what is the partial order?



Packet i. generation time. arrival time, delivery time

S_i C_i D_i

$$0 \le S_1 \le S_2 \le \cdots$$

$$S_i \le C_i \le D_i.$$

S_i and C_i are arbitrarily given.

In-order arrivals: $S_i \le S_{i+1}$, $C_i \le C_{i+1}$.

Out-of-order arrivals: $S_i \le S_{i+1}$, $C_i > C_{i+1}$,

B: buffer size.

M: No. of servers.

If $B=0$, system can keep M packets,

i.i.d. non-expenential service times,

AoI:

$$\Delta(t) = t - \max\{S_i : D_i \le t\}.$$

New-Better-than-Used (NBU) distribution:

X: the service time of a packet.

$R = [X - t \mid X > t]$: the remaining processing time of a packet after t seconds, given that $X > t$.

$$X \geq_{st} R$$

$$\Pr(X > s) \geq \Pr(R > s)$$

$$= \Pr(X - t > s \mid X > t),$$

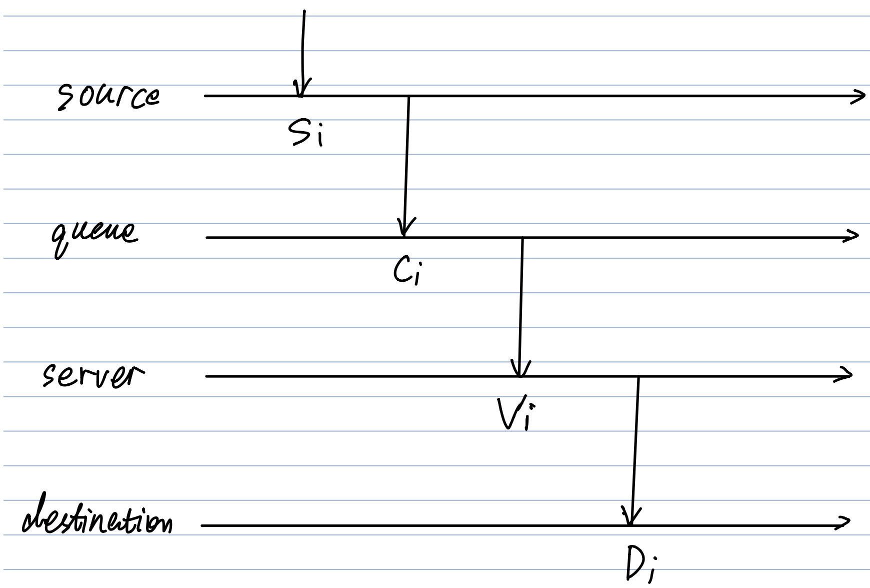
$$= \Pr(X > s + t \mid X > t)$$

$$= \frac{\Pr(X > s+t)}{\Pr(X > t)}.$$

$$\Pr(X > s+t) \leq \Pr(X > s)\Pr(X > t) \quad \forall\, s, t \geq 0,$$

Examples of NBU distributions:

1. exp distribution.
2. const distribution.
3. const + exp
4. Geo distribution.
5. gamma distribution.
   Erlang



Def: V_i: service starting time of packet i.

$$S_i \le C_i \le V_i \le D_i .$$

Note: For a non-preemptive policy,

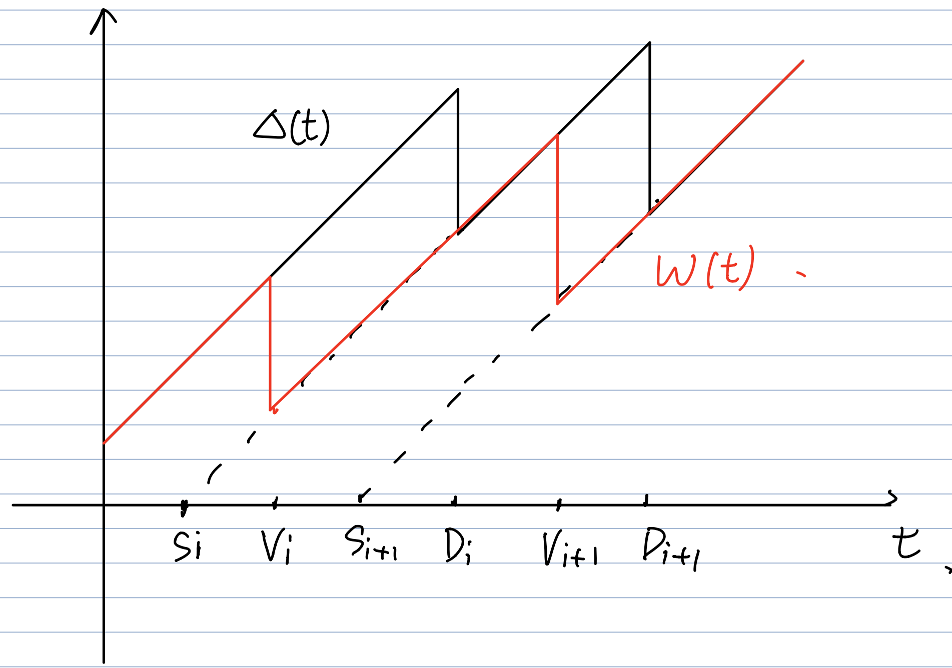
$$D_i - V_i = \text{service time of packet } i.$$

Def: Age of served information $W(t)$,

$$W(t) = t - \max\{S_i : V_i \le t\}.$$

$$\Delta(t) = t - \max\{S_i : D_i \le t\}.$$

$$W(t) \le \Delta(t),$$



Thm:

If (i) the packet service times are i.i.d. NBU.

then for all $M \geq 1$. $B \geq 1$, I, and $\pi \in \Pi_{np}$.

$$[\{W_{non\text{-}prmp\text{-}LGFt}(t),\ t \geq 0\} \mid I] \leq_{st} [\{\Delta_\pi(t), t \geq 0\} \mid I].$$

where Π_{np} is the set of non-preemptive policies.

or. for all non-decreasing functional f.

$$E[f(\{W_{non\text{-}prmp\text{-}LGFt}(t), t \geq 0\} \mid I]$$

$$\leq \min_{\pi \in \Pi_{np}} E[f(\{\Delta_\pi(t), t \geq 0\}) \mid I]$$

$$\leq E[f(\{\Delta_{non\text{-}prmp\text{-}LGFt}(t), t \geq 0\} \mid I]$$

E.g.: $f(\{\Delta(t), t \geq 0\}) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \Delta(t)\, dt$.

Corollary:

Under the conditions of the Thm, we have

$$\min_{\pi \in \Pi_{np}} \limsup_{T \to \infty} \frac{1}{T} E\left[\int_0^T \Delta_\pi(t)\, dt \,\middle|\, I\right]$$

$$\leq \limsup_{T \to \infty} \frac{1}{T} E\left[\int_0^T \Delta_{\text{non-prmp-LGFS}}(t)\, dt \,\middle|\, I\right]$$

$$\leq \min_{\pi \in \Pi_{np}} \limsup_{T \to \infty} \frac{1}{T} E\left[\int_0^T \Delta_\pi(t)\, dt \,\middle|\, I\right] + E[X]$$

mean service time

Proof of Thm:

Section 2.6.2 of the book

Sun, Kadota, Talak, Modiano 2019,

HW3. Problem 2.

Read Section 2.6.2. explain your understanding of

"weak work-efficiency ordering"

and its role in the proof of the Thm.